Behavior of the S Parameter in the Crossover Region Between Walking and QCD-Like Regimes of an SU(N) Gauge Theory

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We consider a vectorial, confining SU(N) gauge theory with a variable number, N_f, of massless fermions transforming according to the fundamental representation. Using the Schwinger-Dyson and Bethe-Salpeter equations, we calculate the S parameter in terms of the current-current correlation functions. We focus on values of N_f such that the theory is in the crossover region between the regimes of walking behavior and QCD-like (non-walking) behavior. Our calculations indicate that the contribution to S from a given fermion decreases as one moves from the QCD-like to the walking regimes. The implications of this result for technicolor theories are discussed.

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I. INTRODUCTION

The properties of a vectorial gauge theory as a function of the fermion content are of fundamental importance. Here we consider such a theory (in $(3+1)$ dimensions at zero temperature and chemical potential) with gauge group SU(N) and N_f massless fermions transforming according to the fundamental representation of this group. For $N = N_c = 3$ and $N_f = 2$, this is an approximation to actual quantum chromodynamics (QCD) with just the u and d quarks, since their current-quark masses are much smaller than the scale $\Lambda_{QCD} \simeq 400$ MeV. We restrict here to the range $N_f < (11/2)N$ for which the theory is asymptotically free. An analysis using the two-loop beta function and Schwinger-Dyson equation (reviewed below) leads to the inference that for N_f in this range, the theory includes two phases: (i) for $0 \le N_f \le N_{f,cr}$ a phase with confinement and spontaneous chiral symmetry breaking (SχSB), and (ii) for $N_{f,cr} \le N_f \le (11/2)N$ a non-Abelian Coulomb phase with no confinement or spontaneous chiral symmetry breaking. We shall refer to $N_{f,cr}$, the critical value of N_f, as the boundary of the non-Abelian Coulomb (conformal) phase [1].

For N_f slightly less than $N_{f,cr}$, the theory exhibits an approximate infrared (IR) fixed point. Let the SU(N) running gauge coupling be denoted as $g(\mu)$, where μ denotes the energy or momentum scale, and let $\alpha(\mu) = \bar{g}(\mu)^2/(4\pi)$. As μ decreases from large values, $\alpha(\mu)$ grows to be O(1) at a scale Λ, but increases only rather slowly as μ decreases below Λ, so that there is an extended interval in energy below Λ where α is large, but slowly running ("walking"). In addition to its intrinsic field-theoretic interest, this walking behavior is an essential ingredient of modern technicolor models of dynamical electroweak symmetry breaking [2], providing the requisite enhancement of Standard-Model (SM) fermion masses [3]-[8]. As N_f approaches $N_{f,cr}$ from below, quantities with dimensions of mass vanish continuously; i.e., the chiral phase transition separating phases (i) and (ii) is continuous.

In this paper, we shall use solutions of the Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations to compute a derivative of the difference of the vector and axial-vector current-current correlation functions, $\Pi'_{VV}(0) - \Pi'_{AA}(0)$. Up to a multiplicative factor, this is the coefficient $\bar{L}_{10}$ of one of the terms in the effective chiral Lagrangian for the theory [9, 10]. Moreover, in the context in which one considers the SU(N) theory as a technicolor (TC) model, with $N = N_{TC}$, the above quantity is proportional to the correction to the Z propagator due to virtual electroweak-nonsinglet technicolor particles, often denoted as S [14, 15], [16, 17]. We focus on the crossover region between the walking regime that occurs for $N_f \lesssim N_{f,cr}$ and the QCD-like (non-walking) regime that occurs for smaller N_f.

There are several motivations for this work. The quantity S, or equivalently $\bar{L}_{10}$, is an intrinsic property of the SU(N) theory, and it is of interest to understand how this quantity depends on N_f. Further, our calculations have important implications for technicolor models of dynamical electroweak symmetry breaking [3]-[8]. For this application, as noted, we identify the SU(N) group with the technicolor gauge group. Precision electroweak data [16, 17] determine allowed regions of values of S and the other Z and W propagator corrections denoted T and U, and yield a stringent constraint on the contributions from new particles in technicolor models. In order to assess the viability of these models, it is necessary to have a reliable calculation of the contribution to S from technicolor particles. A perturbative calculation of S is not reliable since the technifermions are strongly interacting on the relevant scale $\sim m_Z$. Although it is possible to carry out a nonperturbative calculation for technicolor theories that behave like scaled-up QCD (by using QCD data as input for the relevant spectral functions), such theories are excluded since they cannot produce sufficiently large standard-model fermion masses. It is more difficult to carry out a nonperturbative estimate of S for technicolor models that have walking behavior; such estimates have been presented in Refs. [19]-[23]. These suggest that the contribution to S per technifermion electroweak doublet is reduced in the walking region. Meson masses and f_P

(the generalization of f_π) were calculated in this walking regime in Ref. [26]. (See also Ref. [27] for the analogous calculations in QCD.) A motivation for our calculations in this paper is to gain further insight into the behavior of S by studying its behavior in the crossover region between the walking and QCD-like regimes. One of the reasons for concentrating on this crossover region is that the theory still has an approximate infrared fixed point here, and hence one can study the dependence of S on N_f without having to introduce a model-dependent cutoff on the growth of the SU(N) gauge coupling in the infrared that was required in calculations of S via the SD and BS equations for small N_f values, such as $N_f = 3$ in QCD [21, 28]. As our previous calculations of meson masses and f_P in this crossover region showed, [25], although it is a restricted interval in N_f or equivalently, the value of the infrared fixed point, it is sufficient large to observe a significant change between walking and QCD-like behavior.

This paper is organized as follows. Section II is devoted to a review some background material concerning the beta function, approximate infrared fixed point and walking behavior, and technicolor models. Section III contains definitions of the current-current correlation functions and related spectral functions together with the expression for S in terms of these correlation functions and an equivalent integral of spectral functions. In this section we also give the definition of $\bar{L}_{10}$ as a coefficient of a certain operator in the low-energy effective chiral Lagrangian for the theory. Section IV explains how the current correlators are obtained from Bethe-Salpeter amplitudes while Section V discusses the method of solution of the Bethe-Salpeter equation. Our results are presented in Section VI and their implications for technicolor theories in Section VII. An appendix discusses some analytic approximations.

II. SOME PRELIMINARIES

In this section we review some background for our calculations. For the theory under consideration, with an SU(N) gauge group and N_f massless fermions in the fundamental representation, the renormalization group (RG) equation for the running coupling $\alpha(\mu)$ is

$$\beta = \mu \frac{d\alpha(\mu)}{d\mu} = -\frac{\alpha(\mu)^2}{2\pi} \left(b_0 + \frac{b_1}{4\pi}\alpha + O(\alpha^2) \right) , \quad (2.1)$$

where μ is the momentum scale. The two terms listed are scheme-independent. The next two higher-order terms have also been calculated but are scheme-dependent; their inclusion does not significantly affect our results. For the relevant case of an asymptotically free theory, $b_0 > 0$ so that an infrared fixed point exists if and only if $b_1 < 0$. This coefficient b_1 is positive for $0 \le N_f \le N_{f,IR}$, where $N_{f,IR} = (34N^3)/(13N^2 - 3)$, and negative for larger N_f. For $N = 3$, $N_{f,IR} \simeq 8.1$ [29]. The value of α at this IR fixed point, denoted α_*, is given by $\alpha_* = -4\pi b_0/b_1$. Substituting the known values of these terms [30, 31], one has

$$\alpha_* = \frac{-4\pi(11N - 2N_f)}{34N^2 - 13NN_f + 3N^{-1}N_f} \ . \quad (2.2)$$

Solving Eq. (2.2) for N_f in terms of α_* yields

$$N_f = \frac{2N^2[17N(\alpha_*/\pi) + 22]}{(13N^2 - 3)(\alpha_*/\pi) + 8N} \ . \quad (2.3)$$

As is evident from Eqs. (2.2) and (2.3), α_* is a monotonically decreasing function of N_f and equivalently N_f is a monotonically decreasing function of α_*, for $N_{f,IR} \le N_f \le (11/2)N$.

To study the dependence of S on N_f, what we actually vary is the value of the approximate IR fixed point α_*, which depends parametrically on N_f. For definiteness, we shall take $N = 3$; however, as will be seen, N only enters indirectly, via the dependence of the value of the infrared fixed point α_* (Eq. (2.6) below) on N_c. Hence, our findings may also be applied in a straightforward way, with appropriate changes in the value of α_*, to an SU(N) gauge theory with a different value of N.

In the one-gluon exchange approximation, the Schwinger-Dyson gap equation for the inverse propagator of a fermion transforming according to the representation R of SU(N) has a nonzero solution for the dynamically generated fermion mass, which is an order parameter for spontaneous chiral symmetry breaking, if $\alpha \ge \alpha_{cr}$, where α_{cr} is given by

$$\frac{3\alpha_{cr}C_2(R)}{\pi} = 1, \quad (2.4)$$

and $C_2(R)$ denotes the quadratic Casimir invariant for the representation R [32]. Using

$$C_2(fund.) \equiv C_{2f} = \frac{N^2 - 1}{2N} \quad (2.5)$$

for the fundamental representation yields

$$\alpha_{cr} = \frac{2\pi N}{3(N^2 - 1)} \ . \quad (2.6)$$

For the case $N = 3$ that we use for definiteness here, Eq. (2.6) gives $\alpha_{cr} = \pi/4 \simeq 0.79$. To estimate $N_{f,cr}$, one solves the equation $\alpha_* = \alpha_{cr}$, yielding the result [8]

$$N_{f,cr} = \frac{2N(50N^2 - 33)}{5(5N^2 - 3)} \ . \quad (2.7)$$

For the values $N = 3$ and $N = 2$ this gives $N_{f,cr} \simeq 11.9$ and $N_{f,cr} \simeq 7.9$, respectively. These estimates are only rough, in view of the strongly coupled nature of the physics. Effects of higher-order gluon exchanges have been studied in Ref. [33]. In principle, lattice gauge simulations should provide a way to determine $N_{f,cr}$, but the groups that have studied this have not reached a consensus [34].

If α_* is less than the critical value, α_{cr}, for a bilinear fermion condensate to form, the above IR fixed point is exact, with the coupling α approaching α_* from below as the energy scale E decreases from large values to zero. Let us denote the fermions as f_i^a with $a = 1, ..., N$ and $i = 1, ..., N_f$. If $\alpha_* > \alpha_{cr}$, as E descends from large values, the coupling α eventually exceeds the above critical value, the fermion condensates $\langle \sum_{a=1}^{N} \bar{f}_{a,i} f_i^a \rangle$ (with no sum on i) form, and are equal for each flavor $i = 1, ..., N_f$ (with electroweak interactions negligibly small relative to the SU(N) interaction). Accordingly, the global $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V$ symmetry (where $\mathrm{U}(1)_V$ is fermion number) is broken to generalized isospin times fermion number, $\mathrm{SU}(N_f)_V \times U(1)_V$.

Associated with this, the fermions pick up dynamical masses Σ and are integrated out as the energy scale decreases below Σ. Hence, in this case, the IR fixed point is only approximate since in the effective field theory for energies below Σ, the form of the beta function is that for the pure gauge theory, $N_f = 0$. The case where α_* is close to, and slightly larger than, α_{cr}, yields walking behavior. In the strong walking regime, the dynamical fermion mass, and also hadron masses are exponentially smaller than the scale Λ at which the coupling first becomes O(1) as the energy scale decreases from large values. Although our SD and BS equations are semi-perturbative, the analysis is self-consistent in the sense that our α_{cr} really is the value at which, in our approximation, one passes from the confinement phase to the non-Abelian Coulomb phase, and our values of α do span the interval over which there is a crossover from walking to QCD-like (i.e., non-walking) behavior.

As is evident from the above results, decreasing N_f below $N_{f,cr}$ has the effect of increasing α_* and thus moving the theory deeper in the phase with confinement and spontaneous chiral symmetry breaking, away from the boundary with the non-Abelian Coulomb phase. This is the key parametric dependence that we shall use for our study. In Refs. [24, 26] the range of α_* used for the calculation of meson masses was chosen to be $0.89 \le \alpha_* \le 1.0$, an interval where there is pronounced walking behavior. For the case $N = 3$ considered in Ref. [24] and here, given the above-mentioned value, $\alpha_{cr} = \pi/4$, it follows that this lower limit, $\alpha_* = 0.89$, is about 12 % greater than this critical coupling. The reason for this choice of lower limit on α_* was that the calculation of S involves very strong cancellations as $\alpha_* - \alpha_{cr} \to 0^+$, rendering it progressively more and more difficult to obtain accurate numerical results in this extreme walking limit. For our present study of S we consider an interval extending to larger couplings, from $\alpha_* = 1.0$ to $\alpha_* = 1.8$. Our upper limit is chosen in order for the ladder approximation used in our solutions of the Schwinger-Dyson and Bethe-Salpeter equations to have reasonable reliability. From Eq. (2.3) it follows that $\alpha_* = 0.89$ corresponds to $N_f = 11.65$, about 2 % less than $N_{f,cr}$. For a coupling as large as $\alpha_* = 1.8$, the semi-perturbative methods used to derive eqs. (2.2) and (2.3) are subject to large corrections from higher-order perturbative, and from non-perturbative, contributions; recognizing this, the above upper limit of α_* corresponds to $N_f \simeq 10.3$, a roughly 13 % reduction from $N_{f,cr} = 11.9$. Although this shift appears to be by only a modest amount when expressed in terms of N_f, in terms of α_* it is a factor of two, and our calculations in Ref. [25] showed a dramatic change in the values of meson mass ratios and f_P/Λ in this range, with these values changing from their walking limits toward QCD-like values. Hence we anticipate that this range can be sufficient to study the shift in the value of S, and our results confirm this.

If $N_f > N_{f,cr}$, i.e., $\alpha_* < \alpha_{cr}$ so that this IR fixed point of the two-loop RG equation is exact, then, denoting $b \equiv b_0/(2\pi)$, the solution to this equation can be explicitly written [35, 36] in the entire energy region as

$$\alpha(\mu) = \alpha_* \left[\, W(e^{-1}(\mu/\Lambda)^{b\alpha_*}) + 1 \, \right]^{-1}, \tag{2.8}$$

where $W(x) = F^{-1}(x)$, with $F(x) = xe^x$, is the Lambert W function, and Λ is a RG-invariant scale defined by [6]

$$\Lambda \;\equiv\; \mu \, \exp\left[-\frac{1}{b} \left\{ \frac{1}{\alpha_*} \ln\left(\frac{\alpha_* - \alpha(\mu)}{\alpha(\mu)} \right) + \frac{1}{\alpha(\mu)} \right\} \right]. \tag{2.9}$$

Now since we are studying the confined phase with $N_f < N_{f,cr}$, ($\alpha_* > \alpha_{cr}$) with spontaneous chiral symmetry breaking, α_* is only an approximate, rather than exact, IR fixed point. Hence, the solution (2.8) is only applicable in an approximate manner to our case; for momenta much less than the dynamical fermion mass Σ, the fermions decouple, and in this very low-momentum region, with the fermions integrated out, the resultant α would increase above the value α_* at the approximate IR fixed point. However, since $\Sigma \ll \Lambda$ in a walking or near-walking theory, it follows that this lowest range of momenta makes a small contribution to the relevant integrals to be evaluated in our calculations. Hence, over most of the integration range for these integrals where the coupling α is large, it is approximately constant and equal to its fixed-point value, α_* (see Fig. 2 of Ref. [26]). This means that one can use, as a reasonable approximation, the expression

$$\alpha(\mu) = \alpha_* \, \theta(\Lambda - \mu), \tag{2.10}$$

where θ is the step function. (This is the same approximation used in Refs. [25, 26].) Thus, in the walking region and the adjacent crossover region, the calculations have the advantage that one can avoid having to introduce an artificial cutoff on the growth of α in the infrared, in contrast to the situation for smaller N_f, where the walking behavior disappears and this cutoff is necessary.

Since an important application of our results is to technicolor models, we briefly mention some relevant features of these models. As noted, the technicolor gauge theory has a gauge group SU(N_{TC}) and an asysmptotically free coupling that gets large at the TeV scale [2]. It contains a set of massless, vectorially coupled technifermions. The left-handed components of these fermions transform as doublets under $\mathrm{SU}(2)_L$. The spontaneous chiral symmetry breaking and formation of a bilinear technifermion condensate breaks the electroweak symmetry from $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ to $\mathrm{U}(1)_{em}$, producing masses for the W and Z given to leading order by $m_W^2 = m_Z^2 \cos^2\theta_W = (g^2/4) N_{D,TF} F_{TC}^2$, where F_{TC} is the technicolor analogue of f_π. In order to give masses to the Standard-Model fermions (which are technisinglets), it is necessary to embed technicolor in a larger, extended technicolor (ETC) theory [37, 38], with interactions that transform technifermions to the Standard-Model fermions and vice versa. To satisfy constraints from flavor-changing neutral-current (FCNC) processes, the ETC vector bosons that mediate generation-changing transitions must have large masses, ranging from a few TeV to 10^3 TeV. For our present study, concerned with S, we concentrate on the technicolor theory at the scale of a few hundred GeV, with the ETC gauge bosons integrated out.

We focus here on models in which the technifermions transform according to the fundamental representation of the $\mathrm{SU}(N_{TC})$ gauge group. Two simple examples are the so-called one-doublet and one-family technicolor models. A one-doublet TC model has $N_f = 2$ technifermions, denoted U and D, whose chiral components transform according to

$$\begin{aligned} F_L = \begin{pmatrix} U_L \\ D_L \end{pmatrix} \ &: \ (N_{TC}, 1, 2)_{0,L}, \\ U_R \ &: \ (N_{TC}, 1, 1)_{1,R}, \\ D_R \ &: \ (N_{TC}, 1, 1)_{-1,R} \ . \end{aligned} \tag{2.11}$$

where the numbers in parentheses refer to the dimensions of the representations of $\mathrm{SU}(N_{TC}) \times \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L$ and the subscripts refer to the weak $\mathrm{U}(1)_Y$ hypercharges. The value $N_{TC} = 2$ has been preferred in recent TC/ETC model-building [41, 42] for several reasons, including the fact that it (i) minimizes technicolor contributions to the S parameter, (ii) can naturally produce a walking theory in a one-family model (see below), and (iii) makes possible a mechanism to explain light neutrino masses [42]. The one-doublet TC model has one $\mathrm{SU}(2)_L$ doublet of technifermions for each technicolor index, which we express as $N_{D,TF} = 1$, and hence a total number of $\mathrm{SU}(2)_L$ doublets of technifermions equal to $N_{D,tot} = N_{D,TF} N_{TC} = N_{TC}$. The TC sector with just these $N_f = 2$ technifermions would not exhibit walking behavior, but one can add SM-singlet technifermions to produce a theory that does have such behavior [45]. In a one-family TC model the technifermions transform as

$$\begin{aligned} Q_L &: \ (N, 3, 2)_{1/3,L} \\ u_R &: \ (N, 3, 1)_{4/3,R} \\ d_R &: \ (N, 3, 1)_{-2/3,R} \\ L_L &: \ (N, 1, 2)_{-1,L} \\ N_R &: \ (N, 1, 1)_{0,R} \\ E_R &: \ (N, 1, 1)_{-2,R} \end{aligned} \tag{2.12}$$

Hence, this type of technicolor models contains $N_f = 2(N_c + 1) = 8$ technifermions. As is evident from Eq. (2.7), with $N = N_{TC} = 2$, the value $N_f = 8$ is close to the value $N_{f,cr}$ and hence, to within the accuracy of the two-loop beta function analysis, this technicolor model can naturally exhibit walking behavior. Reverting to general $N = N_{TC}$ for our discussion, this one-family technicolor model thus has $N_{D,TF} = (N_c + 1) = 4$ $\mathrm{SU}(2)_L$ doublets for each technicolor index, and hence a total of $N_{D,tot.} = 4N_{TC}$ $\mathrm{SU}(2)_L$ doublets of technifermions.

Because of the spontaneous chiral symmetry breaking in the technicolor theory, the technifermions pick up dynamical masses Σ_{TC} proportional to $F_{TC} N_{TC}^{-1/2}$, where we have included the N_{TC}-dependent factor that would be present in the large-N_{TC} limit, since f_P and Σ_{TC} scale, respectively, like $N_{TC}^{1/2}$ and N_{TC}^0 in this limit. For the one-doublet and one-family TC models, $F_{TC} \simeq 250$ and 125 GeV, respectively. In QCD, the constituent quark mass $\Sigma \simeq 3.5 f_\pi$, and one expects a roughly similar ratio in TC theories (see Fig. 3 in our previous work [25]). Since the SM gauge couplings are small at the technicolor scale, different technifermions are expected to have roughly degenerate dynamical masses, and the contributions of the techniquark and technilepton doublets to one-loop corrections to the Z propagator are approximately equal.

III. EXPRESSION FOR S IN TERMS OF CURRENT-CURRENT CORRELATION FUNCTIONS

As a measure of corrections to the Z propagator arising from heavy particles in theories beyond the Standard Model, S was originally defined as [14]

$$S = \frac{4 s_W^2 c_W^2}{\alpha_{em}(m_Z)} \left. \frac{d\Pi_{ZZ}^{(NP)}(q^2)}{dq^2} \right|_{q^2=0} \tag{3.1}$$

where $s_W^2 = 1 - c_W^2 = \sin^2\theta_W$, evaluated at m_Z and the superscript NP refers to the fact that the definition includes new physics beyond the Standard Model. In the case of technicolor, the technifermions are taken to have zero masses; because of the spontaneous chiral symmetry breaking in the TC theory, they pick up dynamical masses Σ_{TC} of order the technicolor scale. More recent analyses of precision electroweak data define S slightly differently, replacing the derivative at $q^2 = 0$ by a discrete difference (in the $\overline{MS}$ scheme) [16]

$$S_{PDG} = \frac{4 s_W^2 c_W^2}{\alpha_{em}(m_Z)} \left[\frac{\Pi_{ZZ}^{(NP)}(m_Z^2) - \Pi_{ZZ}^{(NP)}(0)}{m_Z^2} \right] . \tag{3.2}$$

The difference between these definitions is small if the heavy physics scale Σ_{TC} satisfies $(2\Sigma_{TC}/m_Z)^2 \gg 1$, as is the case in the TC models considered here.

For our purposes it will be convenient to use the original definition, Eq. (3.1). The implications of our results for technicolor theories would be essentially the same if we used the expression (3.2). With either definition, since a one-loop heavy fermion correction to the Z propagator has a prefactor $(g^2+g'^2)/(16\pi^2)$, where g and g' are the respective $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings, and since $(g^2+g'^2)/(16\pi^2) = \alpha_{em}/(4\pi s_W^2 c_W^2)$, the prefactor $4s_W^2 c_W^2/\alpha_{em}(m_Z)$ in the definition of S cancels out the leading dependence on the electroweak gauge couplings (evaluated at the scale m_Z), yielding a quantity that depends on the intrinsic properties of the strongly coupled $\mathrm{SU}(N)$ gauge theory.

Now, suppressing the $\mathrm{SU}(N_{TC})$ gauge index, we write the fermions as a vector, $\psi = (\psi_i, ..., \psi_{N_f})$. We then define vector and the axial-vector currents as

$$\begin{aligned} V_\mu^a(x) &= \bar{\psi}(x) T^a \gamma_\mu \psi(x) \\ A_\mu^a(x) &= \bar{\psi}(x) T^a \gamma_\mu \gamma_5 \psi(x)\,, \end{aligned} \tag{3.3}$$

where the $N_f \times N_f$ matrices T^a ($a = 1, ..., N_f^2 - 1$) are the generators of $SU(N_f)$ with the standard normalization $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. In terms of these currents, the two-point current-current correlation functions Π_{VV} and Π_{AA} are defined via the equations

$$\begin{aligned} & i \int d^4x \; e^{iq\cdot x} \; \langle 0 | T(J_\mu^a(x) J_\nu^b(0)) | 0 \rangle \\ &= \delta^{ab} \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) \Pi_{JJ}(q^2)\;, \end{aligned} \tag{3.4}$$

where $J_\mu^a(x) = V_\mu^a(x), A_\mu^a(x)$. With the above normalization of T^a, $\Pi(q^2)$ measures the contributions to the time-ordered product in Eq. (3.4) per fermion. Given that, to a good approximation, different technifermion doublets contribute equally to S, it is natural to define a reduced quantity, $\hat{S}$, that represents the contribution to S from each such pair, viz.,

$$\hat{S} = \frac{S}{N_D}\;. \tag{3.5}$$

Then, in terms of the current-current correlation functions defined above, S, as defined in Eq. (3.1), is given by

$$\hat{S} = 4\pi \, \frac{d}{dq^2} \left[\Pi_{VV}(q^2) - \Pi_{AA}(q^2) \right] \Big|_{q^2=0}\;, \tag{3.6}$$

It is convenient to define the compact notation

$$\Pi_{V-A}(q^2) \equiv \Pi_{VV}(q^2) - \Pi_{AA}(q^2)\;. \tag{3.7}$$

As is evident from Eq. (3.6), one may also consider S in a different context, namely that of an abstract vectorial $\mathrm{SU}(N)$ gauge theory with N_f massless fermions transforming according to the fundamental representation of this group, and with all other interactions much weaker in strength than the $\mathrm{SU}(N)$ gauge interaction. In this case, in contrast to technicolor models, where N_f is even (since the technifermions have left-handed components forming $\mathrm{SU}(2)_L$ doublets), N_f can be even or odd (being restricted to be less than $N_{f,cr}$ so that the theory is in the confinement phase with spontaneous chiral symmetry breaking). Here, one could naturally define the contribution to S from each fermion individually, namely, S/N_f. However, since our main application will be to technicolor, we shall continue, as in Ref. [24], to present our results in terms of $\hat{S}$.

Because of the asymptotic freedom of the $\mathrm{SU}(N)$ theory, for Euclidean q^2 much larger than Λ^2, dimensional considerations imply that, asymptotically, $\Pi_{V-A}(q^2) \simeq \langle\bar{\psi}\psi\rangle^2/q^4$ up to logs arising from anomalous dimensions. Combining this property with the analytic properties of $\Pi_{V-A}(q^2)$, one can write the following dispersion relation, with $t \equiv q^2$:

$$\Pi_{V-A}(t) = \frac{1}{\pi} \int_0^\infty ds \, \frac{Im(\Pi_{V-A}(s))}{s - t - i\epsilon} \tag{3.8}$$

The dimensionless spectral function $\rho_J(s)$ is defined as

$$\rho_J(s) \equiv \frac{Im(\Pi_{JJ}(s))}{\pi s} \tag{3.9}$$

This spectral function encodes information about the hadronic states produced by the current J. In terms of these spectral functions, one then has

$$\hat{S} = 4\pi \int_0^\infty \frac{ds}{s} \Big[\rho_V(s) - \rho_A(s) \Big] \tag{3.10}$$

An early discussion of the integral on the right-hand side of Eq. (3.10) and its connection to $\pi^+ \to \ell^+ \nu_\ell \gamma$ decay appeared in [46]. It is of interest to comment on the two equivalent expressions, Eqs. (3.6) and (3.10) for $\hat{S}$. The first of these, Eq. (3.6), expresses $\hat{S}$ in terms of the slope of $\Pi_{V-A}(q^2)$ at $q^2 = 0$. In contrast, the second, Eq. (3.10) expresses it as an integral of $1/s$ times the difference of the physical spectral functions for the vector and axial vector currents in the timelike region, which depends on the spectrum of hadronic states and the strengths of their couplings to these currents. Thus, naively, one might think that these two expressions depend on rather different properties of the $\mathrm{SU}(N)$ theory. The fact that they are actually equivalent is a consequence of the analytic properties of $\Pi_{JJ}(q^2)$ which are used, via the Cauchy theorem, to derive the resultant dispersion relation. Thus, although it is evaluated at a single point, the derivative $\Pi'_{V-A}(0)$ "knows" about the full hadronic spectrum, including the presence or absence of walking.

In passing, we note that the spectral functions directly in terms of the products of currents, as

$$i \int d^4x \; e^{iq\cdot x} \; \langle 0 | T(V_\mu^a(x) V_\nu^b(0)) | 0 \rangle$$

$$= -\delta^{ab} \int_0^\infty ds \, \frac{\rho_V(s)(sg_{\mu\nu} - q_\mu q_\nu)}{s - q^2 - i\epsilon} \qquad (3.11)$$

and

$$i \int d^4x \; e^{iq\cdot x} \; \langle 0|T(A^a_\mu(x)A^b_\nu(0))|0\rangle$$
$$= -\delta^{ab}\Bigg[\int_0^\infty ds \, \frac{\rho_A(s)(sg_{\mu\nu} - q_\mu q_\nu)}{s - q^2 - i\epsilon} + \frac{q_\mu q_\nu}{q^2} f_P^2 \Bigg] , \qquad (3.12)$$

where one explicitly separates out the contribution from the ($J = 0$ part of) $\rho_A(s)$ at $s = 0$ due to the Nambu-Goldstone bosons (NGB's). In Eq. (3.12), the quantity f_P (where P stands for "pseudoscalar") is the N_f-flavor generalization of f_π defined by the transition matrix element

$$\langle 0|A^a_\mu(0)|\pi^b(q)\rangle = iq_\mu f_P \delta^{ab} , \qquad (3.13)$$

with $a, b = 1, 2, \ldots, N_f^2 - 1$. In actual QCD, the chiral symmetry is explicitly broken by the u and d current-quark masses (and also by electroweak interactions), so that the pions decay, and, in particular, the dominant weak decay of the π^+, $\pi^+ \to \mu^+\nu_\mu$ has a rate proportional to f_π^2. Thus, f_P might be called the generalized Nambu-Goldstone decay constant, but we will avoid this term, since in our basic SU(N) theory with other interactions turned off, these Nambu-Goldstone bosons are exactly massless and do not decay. In the chiral limit of QCD, with $m_u = m_d = 0$, it has been estimated that $(f_\pi)_{ch.lim.}/f_\pi \simeq 0.935$ [9], so that, with the physical value $f_\pi = 92.4 \pm 0.3$, one infers that $(f_\pi)_{ch.lim.} \simeq 86$ MeV (with a theoretical uncertainty of several per cent from the chiral extrapolation). This slight decrease will not be important for our work.

The constant f_P may be calculated by first solving the Schwinger-Dyson equation for the momentum-dependent dynamical fermion mass $\Sigma(p)$ and then substituting this into the Pagels-Stokar relation [47],

$$f_P^2 \; = \; \frac{N_c}{4\pi^2} \int_0^\infty y\,dy \, \frac{\Sigma^2(y) \; - \; \frac{y}{4}\,\frac{d}{dy}\left[\Sigma^2(y)\right]}{[y \; + \; \Sigma^2(y) \;]^2} \; . \qquad (3.14)$$

Calculations using this method [21, 25, 26] have shown (for $N = N_c = 3$) that as N_f increases from the value $N_f = 2$ toward $N_{f,cr}$, the generalized quantity f_P decreases, as is expected, since f_P is an order parameter for spontaneous chiral symmetry breaking. Furthermore, in the strong walking limit $N_f \nearrow N_{f,cr}$, i.e., $\alpha_* \searrow \alpha_{cr}$, it has been found that f_P vanishes like [7, 48]

$$f_P = c_f \Lambda \, \exp\left[-\pi\Big(\frac{\alpha_*}{\alpha_{cr}} - 1\Big)^{-1/2}\right] , \qquad (3.15)$$

where c_f is a constant. In Ref. [25] we calculated f_P in the crossover region between this extreme walking limit and smaller values of N_f, corresponding to larger values of α_*, closer to QCD with $N_f = 2$ or 3 and found a dramatic growth in f_P/Λ, approaching values nearer to QCD, as expected.

Because of the asymptotic freedom and consequent asymptotic decay of $\Pi_{V-A}(q^2)$ for large Euclidean q^2 and the fact that the fermions are massless in the underlying SU(N) theory, the following integral relations (first and second Weinberg sum rules) hold [49]-[52]:

$$\int_0^\infty ds \, \Big[\rho_V(s) - \rho_A(s)\Big] = \Pi_{V-A}(0) = f_P^2 \qquad (3.16)$$

and

$$\int_0^\infty ds \, s \, \Big[\rho_V(s) - \rho_A(s)\Big] = 0 \; . \qquad (3.17)$$

Clearly, Eqs. (3.10), (3.16), and (3.17) are of similar form, viz., integrals of $\rho_V(s) - \rho_A(s)$ weighted, respectively by s^{-1}, 1, and s. Because of these different factors, the contributions to the integral (3.10) for $\hat{S}$ are weighted toward smaller s values, while the contributions to the integral (3.17) are weighted toward larger s values, as compared with those for the integral in Eq. (3.16).

Finally, in the context of an abstract SU(N) gauge theory (with other interactions turned off), it is worthwhile to mention the connection between S and the coefficient denoted $\bar{L}_{10}$. To explain this connection, we recall the definition of $\bar{L}_{10}$. Provided that $N_f \geq 2$, the theory contains $N_f^2 - 1$ massless Nambu-Goldstone bosons. (For this discussion, we turn off electroweak interactions completely; for our application to technicolor, we do not turn them off, and hence three of the would-be Nambu-Goldstone bosons become the longitudinal modes of the $W^\pm$ and Z. Furthermore, in the TC/ETC context, other would-be NGB's gain masses from color and ETC interactions that explicitly break the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ global chiral symmetry.) A useful description of the low-energy physics for energies $E \ll 4\pi f_P$ is then provided by a chiral Lagrangian (see. e.g., [9] and references to earlier work therein). This is a function of the chiral fields

$$U = \exp\left(\frac{2i\sum_{a=1}^{N_f^2-1}\pi^a T^a}{f_P}\right) \; . \qquad (3.18)$$

For example, for QCD with just the two light quarks u and d, U would have the form $U = e^{i\pi\cdot\tau/f_\pi}$. Now let us denote the elements of the global symmetry groups $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$ as U_L and U_R. Then under a transformation by an element of the chiral symmetry group $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, $U \to U_L U U_R^\dagger$. One can also formally introduce external gauge fields (contracted with the respective generators of $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$), L_μ and R_μ. These have the transformation properties $L_\mu \to U_L L_\mu U_L^\dagger - i(\partial_\mu U_L)U_L^\dagger$ and $R_\mu \to U_R R_\mu U_R^\dagger - i(\partial_\mu U_R)U_R^\dagger$. In terms of these, one constructs the covariant derivative $D_\mu U = \partial_\mu U - iL_\mu U + iUR_\mu$. One also defines external field strengths $(F_L)_{\mu\nu}$ and $(F_R)_{\mu\nu}$ in the usual manner. The lowest-order term in this effective Lagrangian is then

$$\frac{f_P^2}{4}\,\mathrm{Tr}\left[(D_\mu U)^\dagger (D^\mu U)\right] \ . \tag{3.19}$$

In terms of these quantities, $L_{10}^{(r)}$ is the coefficient of the term

$$O_{10} = \mathrm{Tr}(U^\dagger (F_L)_{\mu\nu} U (F_R)^{\mu\nu}), \tag{3.20}$$

where the superscript (r) refers to the renormalized quantity. From this, one obtains a quantity which is constructed to be independent of the renormalization scale μ [9, 10]

$$\bar{L}_{10} = L_{10}^{(r)}(\mu) + \frac{1}{192\pi^2}\left[\ln\left(\frac{m_\pi^2}{\mu^2}\right) + 1\right] \tag{3.21}$$

The relation with S is then

$$S = -16\pi \bar{L}_{10} \ . \tag{3.22}$$

Note that although $\bar{L}_{10}$ requires $N_f \geq 2$ to be well-defined, it does not require N_f to be even. For the specific case of QCD, fits to experimental data, including, in particular, the radiative decay $\pi^+ \to e^+ \nu_e \gamma$, yield the value [9, 13]

$$S = 0.33 \pm 0.04 \ . \tag{3.23}$$

To the extent that this is dominated by the two light quarks u and d, one has $N_f = 2$, and hence $N_D = 1$, so that the measured value of S for QCD is also equal to $\hat{S}$. The fact that the light-quark vector and axial-vector mesons ρ and a_1 largely saturate the expression for S, Eq. (3.10), is consistent with this conclusion. An approximate calculation of $\hat{S}$ has been performed using the ladder approximation to the Schwinger-Dyson and (inhomogeneous) Bethe-Salpeter equations for QCD ($N = 3$) with $N_f = 2$ quarks of negligible mass [21]. Studies have also been done for the case where one neglects the strange quark mass m_s, i.e., $N = 3$, $N_f = 3$ [21, 28]. Since for either of these values of N_f the beta function of the QCD theory does not exhibit an infrared fixed point, it is necessary to cut off the growth of the strong coupling. For typical cutoffs, it was found that the SD-IBS calculations tended to yield slightly too large a value of $\hat{S} = S$, namely $\hat{S} \simeq 0.45 - 0.5$ [21, 28], rather than a value in the 1σ experimental range $0.29 \lesssim S \lesssim 0.37$. This suggests that in the SD-IBS approach, used for a vectorial confining SU(N) gauge theory with small values of N_f such that the theory has no perturbative IR fixed point, with a typical IR cutoff on the coupling, tends to overestimate S by about a factor of 1.4. Since the calculation of S in QCD is a problem in strongly coupled, nonperturbative physics, and the calculational method that was used is only approximate (neglecting, for example, instanton contributions), one should probably not be surprised that it does not precisely reproduce the measured value of S.

IV. CURRENT-CURRENT CORRELATION FUNCTIONS IN TERMS OF BETHE-SALPETER AMPLITUDES

In this section, we explain how the current-current correlation functions are obtained from the Bethe-Salpeter amplitudes, which will be calculated via the inhomogeneous Bethe-Salpeter (IBS) equation [53]-[63], [48]. These Bethe-Salpeter amplitudes $\chi^{(J)}$, where $J = V$ or A, are essentially form factors, whose behavior in the timelike region describes the coupling of the given current to physical hadronic bound states that can be produced by this current, with an analytic continuation into the spacelike region. Here we will only need these amplitudes for the spacelike region $q^2 < 0$ and at the point $q^2 = 0$. The amplitudes may be defined in terms of the three-point vertex function as

$$\begin{aligned}&\delta_j^k \, (T^a)_f^{f'} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot r}\, \chi_{\alpha\beta}^{(J)}(p;q,\epsilon) = \\ &= \epsilon^\mu \int d^4x\, e^{iq\cdot x} \langle 0|T(\psi_{\alpha f}^k(r/2)\ \bar{\psi}_{jf'\beta}(-r/2)\ J_\mu^a(x))|0\rangle,\end{aligned} \tag{4.1}$$

where (f, f'), (j, k) and (α, β) are, respectively, the flavor, gauge, and spinor indices. Closing the fermion legs of the above three-point vertex function and taking the limit $r \to 0$, one can express the current correlator in terms of the Bethe-Salpeter amplitude as

$$\begin{aligned}&\Pi_{JJ}(q^2) = \\ &\frac{1}{3}\left(\frac{N}{2}\right)\sum_\epsilon \int \frac{d^4p}{i(2\pi)^4}\mathrm{Tr}\left[\left(\epsilon\cdot G^{(J)}\right)\chi^{(J)}(p;q,\epsilon)\right]\end{aligned} \tag{4.2}$$

where [64]

$$G_\mu^{(V)} = \gamma_\mu, \qquad G_\mu^{(A)} = \gamma_\mu\gamma_5, \tag{4.3}$$

and an average has been taken over the polarizations, so that $\Pi_{JJ}(q^2)$ does not depend on the polarization ϵ.

We expand the Bethe-Salpeter amplitude $\chi_{\alpha\beta}^{(J)}(p;q,\epsilon)$ in terms of a complete bispinor basis with basis elements $\Gamma_i^{(J)}$ and the invariant amplitudes $\chi_i^{(J)}$ as

$$\left[\chi^{(J)}(p;q,\epsilon)\right]_{\alpha\beta} = \sum_{i=1}^{8}\left[\Gamma_i^{(J)}(p;\hat{q},\epsilon)\right]_{\alpha\beta}\chi_i^{(J)}(p;q), \tag{4.4}$$

where we define

$$Q^2 \equiv -q^2 \tag{4.5}$$

so that $Q^2 > 0$ in the spacelike region, and set

$$\hat{q}_\mu \equiv \frac{q_\mu}{\sqrt{Q^2}} \tag{4.6}$$

The bispinor basis elements can be chosen in such a manner that they have the same spin, parity and charge conjugation as the corresponding current $J^a_\mu(x)$. For the vector vertex we adopt the following bispinor basis elements:

$$\Gamma_1^{(V)} = \not\epsilon, \;\; \Gamma_2^{(V)} = \frac{1}{2}[\not\epsilon, \not p](p\cdot\hat{q}), \;\; \Gamma_3^{(V)} = \frac{1}{2}[\not\epsilon, \not\hat{q}],$$
$$\Gamma_4^{(V)} = \frac{1}{3!}[\not\epsilon, \not p, \not\hat{q}], \;\; \Gamma_5^{(V)} = (\epsilon\cdot p), \;\; \Gamma_6^{(V)} = \not p(\epsilon\cdot p),$$
$$\Gamma_7^{(V)} = \not\hat{q}(p\cdot\hat{q})(\epsilon\cdot p), \;\; \Gamma_8^{(V)} = \frac{1}{2}[\not p, \not\hat{q}](\epsilon\cdot p), \tag{4.7}$$

where $[a,b,c] \equiv a[b,c] + b[c,a] + c[a,b]$. For the axial-vector vertex we use the bispinor basis elements

$$\Gamma_1^{(A)} = \not\epsilon\;\gamma_5, \; \Gamma_2^{(A)} = \frac{1}{2}[\not\epsilon, \not p]\gamma_5, \; \Gamma_3^{(A)} = \frac{1}{2}[\not\epsilon, \not\hat{q}]\;(p\cdot\hat{q})\;\gamma_5,$$
$$\Gamma_4^{(A)} = \frac{1}{3!}[\not\epsilon, \not p, \not\hat{q}]\gamma_5, \;\; \Gamma_5^{(A)} = (\epsilon\cdot p)\;(p\cdot\hat{q})\;\gamma_5,$$
$$\Gamma_6^{(A)} = \not p(\epsilon\cdot p)\;\gamma_5, \;\; \Gamma_7^{(A)} = \not\hat{q}\;(\epsilon\cdot p)\;(p\cdot\hat{q})\;\gamma_5,$$
$$\Gamma_8^{(A)} = \frac{1}{2}[\not p, \not\hat{q}](\epsilon\cdot p)\;(p\cdot\hat{q})\;\gamma_5. \tag{4.8}$$

Given the charge conjugation properties of the vector and axial-vector currents and the above choice of the bispinor basis elements, it follows that invariant amplitudes $\chi_i^{(J)}$ are even functions of $p\cdot\hat{q}$.

In the present analysis it is convenient to choose the Lorentz reference frame so that only the timelike component of q^μ is nonzero. Since we are working in the spacelike region (which avoids physical mass singularities), we thus use a Wick rotation with

$$q^\mu = (iQ, 0, 0, 0). \tag{4.9}$$

Similarly, for the relative momentum p^μ we perform a Wick rotation and parametrize it in terms of the real variables u and w (with dimensions of mass) as

$$p\cdot q = -Q\;u\;, \;\; p^2 = -u^2 - w^2. \tag{4.10}$$

Hence, the invariant amplitudes $\chi_i^{(J)}$ are functions of u and w:

$$\chi_i^{(J)} = \chi_i^{(J)}(u, w; Q). \tag{4.11}$$

Owing to the charge-conjugation properties of the Bethe-Salpeter amplitude $\chi^{(J)}$ and the bispinor basis elements defined above, the invariant amplitudes $\chi_i^{(J)}(u,w)$ satisfies the relation

$$\chi_i^{(J)}(u, w; Q) = \chi_i^{(J)}(-u, w; Q)\;. \tag{4.12}$$

Using this property of the invariant amplitudes, we rewrite Eq. (4.2) as

$$\Pi_{VV}(Q^2) \;=\; \frac{N}{\pi^3}\int_0^\infty du \int_0^\infty dw\; w^2 \left[-\chi_1^{(V)}(u,w;Q) + \frac{w^2}{3}\chi_6^{(V)}(u,w;Q)\right], \tag{4.13}$$

$$\Pi_{AA}(Q^2) \;=\; \frac{N}{\pi^3}\int_0^\infty du \int_0^\infty dw\; w^2 \left[\;\chi_1^{(A)}(u,w;Q) - \frac{w^2}{3}\chi_6^{(A)}(u,w;Q)\right]. \tag{4.14}$$

Here, we have used the expanded form of the Bethe-Salpeter amplitude given in Eq. (4.4) and carried out the three-dimensional angular integration.

From Eqs. (4.13) and (4.14), it follows that

$$\begin{aligned}\Pi_{V-A}(Q^2) \;&=\; \frac{1}{3}\left(\frac{N}{2}\right)\sum_\epsilon \int \frac{d^4p}{i(2\pi)^4} \\ &\quad \mathrm{Tr}\Big[\not\epsilon\chi^{(J)}(p;q,\epsilon) - \not\epsilon\gamma_5\chi^{(A)}(p;q,\epsilon)\Big] \\ &=\; \frac{N}{\pi^3}\int_0^\infty du \int_0^\infty dw\; w^2 \\ &\quad \Bigg[-\Big(\chi_1^{(V)}(u,w;Q) + \chi_1^{(A)}(u,w;Q)\Big) \\ &\quad +\frac{w^2}{3}\Big(\chi_6^{(V)}(u,w;Q) + \chi_6^{(A)}(u,w;Q)\Big)\Bigg].\end{aligned} \tag{4.15}$$

Although both $\Pi_{VV}(Q^2)$ and $\Pi_{AA}(Q^2)$ are individually logarithmically divergent, the underlying $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry guarantees that these divergences cancel in the difference $\Pi_{VV} - \Pi_{AA}$, which is therefore finite.

V. INHOMOGENEOUS BETHE-SALPETER EQUATION

In this section we discuss the full (inhomogeneous) Bethe-Salpeter equation, which we will use to calculate current-current correlation functions and, from these, S (and, as a check, also f_P). The IBS equation is a self-consistent description of the coupling of a current J^a_μ to fermion-antifermion bound states. This coupling is represented by the Bethe-Salpeter amplitude $\chi^{(J)}$. The four-momenta assigned to the fermion and antifermion are $p_\psi = p + (q/2)$ and $p_{\bar\psi} = -p + (q/2)$ so that the total momentum of the bound state is q, and the relative momentum of the ψ and $\bar\psi$ is $2p$. Since we are dealing with $J = 1$ bound states, the bound-state amplitude also depends on the polarization vector ϵ, which satisfies $\epsilon\cdot q = 0$ and $\epsilon\cdot\epsilon = -1$. A grapical indication of the inhomogeneous Bethe-Salpeter equation structure is given in Fig. 1. The IBS equation is

$$T(p;q)\;\chi^{(J)}(p;q,\epsilon) \;\;=\;\; \epsilon\cdot G^{(J)} \;+\; K(p;k) * \chi^{(J)}(k;q,\epsilon). \tag{5.1}$$

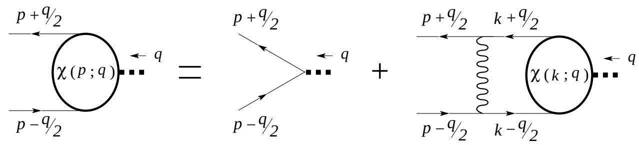

FIG. 1: A graphical expression of the IBS equation in the (improved) ladder approximation.

Here the kinetic part T is

$$T(p;q) = -S_F^{-1}(p+q/2) \otimes S_F^{-1}(p-q/2) \ , \qquad (5.2)$$

where $S_F(p) = 1/(A(p)\not{p} - \Sigma(p))$. We follow the standard procedure of using the Landau gauge in calculations with the Schwinger-Dyson and Bethe-Salpeter equations since in this gauge the fermion wave function renormalization factor $A(p) = 1$. The physical results are, of course, gauge-invariant. The Bethe-Salpeter kernel K in the improved ladder approximation is expressed as

$$\begin{aligned} K(p;k) &= C_{2f} \, \frac{\bar{g}^2(p,k)}{(p-k)^2} \times \\ &\times \left(-g_{\mu\nu} + \frac{(p-k)_\mu (p-k)_\nu}{(p-k)^2} \right) \cdot \gamma^\mu \otimes \gamma^\nu, \quad (5.3) \end{aligned}$$

where C_{2f} was given in Eq. (2.5). In the above expressions we use the tensor product notation

$$(A \otimes B)\, \chi = A \, \chi \, B \ , \qquad (5.4)$$

and the inner product notation

$$K(p;k) * \chi^{(J)}(k;q,\epsilon) = \int \frac{d^4k}{i(2\pi)^4} \ K(p,k) \ \chi(k;q) \ . \quad (5.5)$$

where summations over Dirac indices are understood. (In contrast, for our previous calculations [25, 26] of meson masses, we only needed to use the homogeneous Bethe-Salpeter equation.)

As in Refs. [24, 25, 26, 28, 60], we make the ansatz for the running coupling, after Euclidean rotation,

$$\alpha(p_E, q_E) = \alpha(p_E^2 + q_E^2) \ , \qquad (5.6)$$

where the subscript denotes Euclidean. Since α would naturally depend on the gluon momentum squared, $(p-q)^2 = p^2 + q^2 - 2p\cdot q$, the functional form (5.6) amounts to dropping the scalar product term, $-2p \cdot q$. As discussed in Ref. [25], this is a particularly reasonable approximation in the case of a walking gauge theory because most of the contribution to the integral comes from a region of Euclidean momenta where α is nearly constant. Hence, the shift upward or downward due to the $-2p \cdot q$ term in the argument of α has very little effect on the value of this coupling for the range of momenta that make the most important contribution to the integral. The approximation (5.6) enables one to carry out the angular integration analytically.

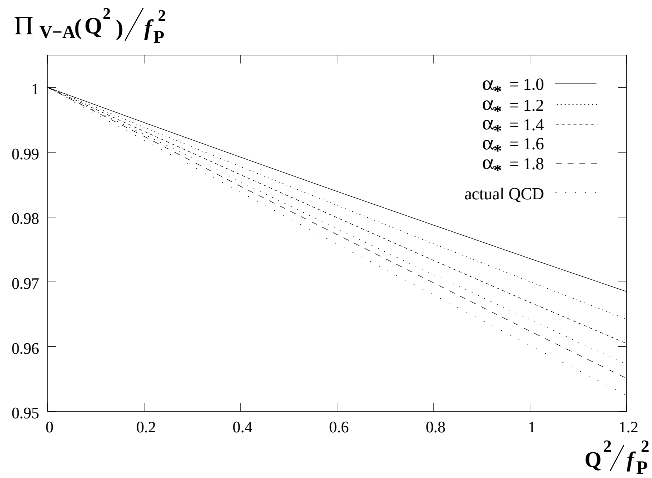

FIG. 2: Plot of $\Pi_{V-A}(Q^2)/f_P^2$ as a function of Q^2 in the spacelike interval $0 \le Q^2/f_P^2 \le 1.2$, for $\alpha_* = 1.0, 1.2, 1.4, 1.6, 1.8$. As indicated, the horizontal axis refers to the quantity Q^2/f_P^2. For comparison, $\Pi_{V-A}(Q^2)/f_P^2$ for QCD with $N_f = 3$ massless quarks is also plotted. See text for further details.

The momentum-dependent dynamical mass $\Sigma(p)$ for the fermion is obtained from the Schwinger-Dyson equation,

$$\Sigma(p) = -K(p,k) * S_F(p). \qquad (5.7)$$

We use the same kernel $K(p,k)$ here as in the IBS equation in order to respect the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry [60, 61]. The numerical method that is used for solving the SD equation and the IBS equation involves approximating the respective integrals by discrete sums and is the same as in Ref. [24, 28]. The reader is referred to these papers for more details on this method.

VI. RESULTS AND DISCUSSION

In this section we present the results of our calculations of $\Pi_{V-A}(Q^2)$ and $\hat{S}$ using the Schwinger-Dyson and (inhomogeneous) Bethe-Salpeter equations. It is appropriate to include an obvious cautionary remark that these calculations involve strong couplings α of order unity, and therefore there could be significant corrections to the (improved) ladder approximation used in our solutions of the Schwinger-Dyson and Bethe-Salpeter equations. Accordingly, as one check on the reliability of our methods, we have also carried out a comparison of f_P obtained from the SD and IBS equations via eq. (3.16) with f_P obtained from the SD equation via the Pagels-Stokar relation.

In Fig.2 we plot our calculated values of $\Pi_{V-A}(Q^2)/f_P^2$ for $\alpha_* = 1.8, 1.6, 1.4, 1.2$ and 1.0, as functions of the dimensionless quantity Q^2/f_P^2. The slope of each curve at $Q^2 = 0$ is equal to $-\hat{S}/(4\pi)$ for the given value of α_*. We first observe that $\Pi_{V-A}(Q^2)$ is almost linear at $Q^2 = 0$,

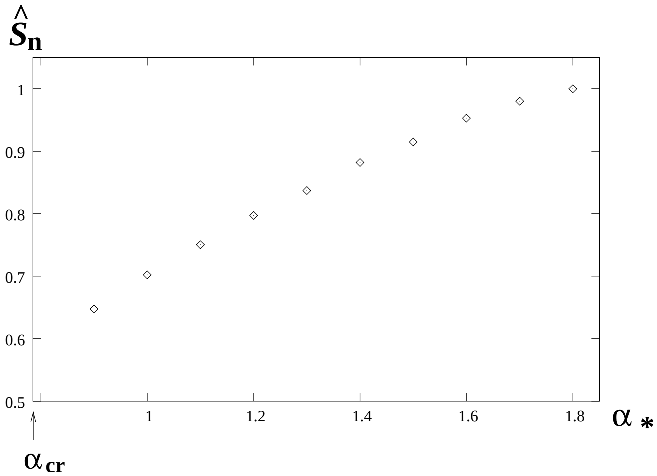

FIG. 3: Plot of $\hat{S}_n$ for several values of α_* in the range $0.9 \le \alpha_* \le 1.8$. As indicated by the subscript n, the values are normalized by the value of $\hat{S}$ at $\alpha_* = 1.8$.

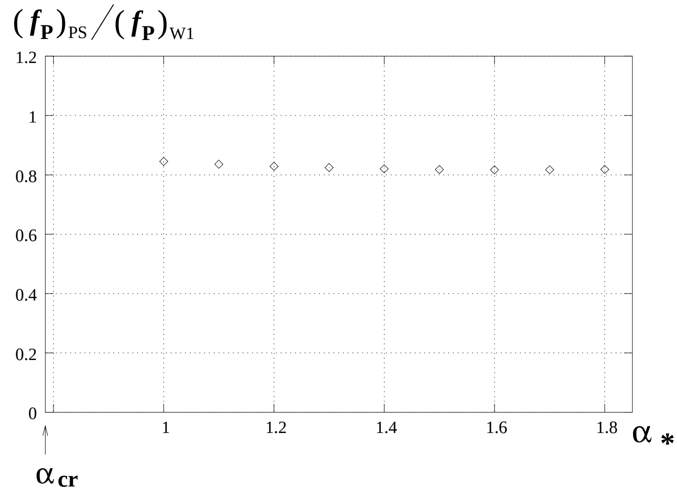

FIG. 4: Plot of the ratio $(f_P)_{PS}/(f_P)_{W1}$, where $(f_P)_{PS}$ and $(f_P)_{W1}$ denote f_P calculated via Eq. (3.14) and via Eq. (3.16), respectively, as a function of α_*.

with a small positive second derivative. This is the justification for our earlier statement that the implications of our findings for technicolor would be essentially the same whether we used the definition of $\hat{S}$ in terms of a derivative at $Q^2 = 0$ or a finite difference, analogously to Eq. (3.2). Secondly, from Fig. 2 it is clear that the magnitude of the slope, and hence $\hat{S}$, decreases as α_* decreases toward the chiral phase transition point α_{cr}.

In Fig. 3, we plot the values of $\hat{S}$, derived from the slope of $\Pi_{V-A}(Q^2)$ at $Q^2 = 0$, for several values of α_*. As indicated by the subscript n, the values are normalized by the value of $\hat{S}$ at $\alpha_* = 1.8$. This figure shows that $\hat{S}$ decreases by about 40 % as α_* is reduced from 1.8 to 0.9, or equivalently (c.f. Eq. (2.3)) as N_f is increased from 10.3 to 11.6. As our calculation of meson masses in Ref. [25] showed, this is a crossover region, in which the theory is changing from QCD-like, non-walking behavior at smaller N_f to the walking regime at larger N_f approaching $N_{f,cr}$. Reinserting the factor of $N_D = N_f/2$ to get S itself, we obtain a decrease by about 30 % in S, since N_D only increases by about 10 % over this range. Thus, our calculation shows that for this range of values, $\hat{S}$ decreases significantly as one moves from the QCD-like to the walking regimes. This finding is an important result of our present study.

As a check on our calculational methods, we compare f_P calculated in two different ways: via the Pagels-Stokar relation, Eq. (3.14), denoted $(f_P)_{PS}$, and by the first Weinberg sum rule (W1) or equivalent relation $\Pi_{V-A}(0) = f_P^2$ in Eq. (3.16), denoted $(f_P)_{W1}$. In view of the fact that the Pagels-Stokar itself is approximate and that our solution of the SD and IBS equations involves the ladder approximation and the neglect of completely nonperturbative contributions such as those due to instantons, we do not expect exact agreement between these two different methods of calculation. In Fig. 4 we present a plot of the ratio $(f_P)_{PS}/(f_P)_{W1}$ as a function of α_*. The closeness of this ratio to unity gives one measure of the accuracy and reliability of our calculations. We see that the ratio is within about 20 % of unity and is essentially independent of α_* in the range considered, with the Pagels-Stokar method yielding a slightly smaller value than the expression in terms of the current-current correlation functions. This gives us further confidence in the results of our SD-IBS calculation of $\hat{S}$.

Our finding that $\hat{S}$ and S are reduced as one moves from the QCD-like regime toward the walking regime of an SU(N) gauge theory is in agreement with the approximate analytic results of Refs. [19, 20, 22, 23], and it complements those works, being based on a numerical solution of the Schwinger-Dyson and Bethe-Salpeter equations. In the sub-interval $0.9 \le \alpha_* \le 1.0$ closer to the walking limit our results coincide with those in Ref. [24]. Our use of a larger interval has the advantage that we are able to observe a larger reduction in $\hat{S}$ as α_* decreases than was done in Ref. [24]. We have not attempted here to examine the extreme walking limit $(\alpha_* - \alpha_{cr})/\alpha_{cr} \to 0^+$. As discussed in Ref. [24], it becomes increasingly difficult to obtain an accurate numerical solution for $\hat{S}$ in this limit because of very strong cancellations.

In addition to this decreasing trend of $\hat{S}$, one may also discuss the absolute magnitude of $\hat{S}$. Our calculation yields $\hat{S} = 0.47$ at $\alpha_* = 1.8$. If $\hat{S}$ continues to be a monotonically function of N_f (and hence, in the range of interest here, also a monotonic function of α_*), then an extrapolation of our calculation to the QCD case with $N_f = 2$ or, with the strange quark mass neglected, $N_f = 3$, would predict a value of $\hat{S} \gtrsim 0.5$. This is similar to the value that was obtained in earlier studies using the SD and IBS equations in a different manner than here, where it was necessary to introduce an cutoff on the growth of the strong coupling in the infrared [21, 28]. In Ref. [28] it was shown that if one used a cutoff that led to a very large value of the coupling, one could get a result for S in agreement with the experimental value (3.23), but the reliability of the calculational method in the presence of such a large coupling was not clear. We shall adopt the optimisitc viewpoint here of giving greater weight to the change in $\hat{S}$ as a function of N_f than to the absolute value of $\hat{S}$ itself. Equivalently, one could envision applying an overall correction factor of about 2/3 to the absolute magnitude of $\hat{S}$ so that the value for small N_f matches that in QCD. Physically, this factor would be regarded as correcting for the strong-coupling effects not included in the SD-IBS analysis.

Although one cannot use perturbation theory reliably to calculate S in a strongly coupled gauge theory, the perturbative result is often used in discussions of constraints on new physics, and hence it is worthwhile to see how our results compare with the perturbative computation. A one-loop perturbative calculation with degenerate fermions having effective masses satisfying $(2\Sigma/m_Z)^2 \gg 1$ yields the well-known result $S_{pert.} = N_{D,tot.}/(6\pi)$ where here $N_{D,tot.} = N_D N$, i.e,

$$\hat{S}_{pert.} = \frac{N}{6\pi} \ . \tag{6.1}$$

In QCD with $N_f = 2$ and $N = N_c = 3$, this perturbative calculation would predict $S_{QCD,pert.} \simeq 1/(2\pi) \simeq 0.16$. The experimental value in Eq. (3.23) is approximately twice this; $S_{QCD} \simeq 2S_{QCD,pert.}$. The reductions that we have found in $\hat{S}$ and S for the range of α_* investigated suggest that in a walking theory, much or all of the above factor of 2 might be removed, and the true value of S might well be comparable to, or, indeed, perhaps less than, the perturbative estimate.

For reference, in the one-doublet and one-family technicolor models, the perturbative expressions for the techniparticle contributions to S are $S_{TC,pert.} = N_{TC}/(6\pi)$ and $S_{TC,pert.} = 2N_{TC}/(3\pi)$, respectively. With $N_{TC} = 2$, these take the values $S_{QCD,pert.} \simeq 0.1$ and 0.4, respectively. Fits to precision electroweak data yield allowed regions in S and two other parameters describing modifications of the Z and W propagators by new physics beyond the Standard Model, namely the parameter T measuring violations of custodial SU(2) from this new physics and a third parameter, U, of somewhat less importance here. Since the Standard Model expression for S includes a term $(1/(6\pi))\ln(m_H/m_{H,ref.})$, the resultant allowed regions depend on the choice of the reference value of the SM Higgs mass, $m_{H,ref.}$. The comparison of these with a technicolor theory is complicated by the fact that technicolor has no fundamental Higgs field; sometimes one formally uses $m_{H,ref.} \sim 1$ TeV for a rough estimate, since the SM with $m_H \sim 1$ TeV has strong longitudinal vector boson scattering, as does technicolor. However, this may involve some double-counting when one also includes contributions to S from technifermions, whose interactions and bound states (e.g., techni-vector mesons) are responsible for the strong $W_L^+ W_L^-$ and $Z_L Z_L$ scattering in a technicolor framework. The current fit [16, 17] disfavors values of $S \gtrsim 0.2$. Our findings in this paper suggest, in agreement with the previous works noted above using different methods, that the constraint on walking technicolor models could be less severe than would be inferred from the perturbative formula for the technifermion contribution to S.

VII. SUMMARY

In summary, using numerical solutions of the Schwinger-Dyson and (inhomogeneous) Bethe-Salpeter equation, we have calculated the S parameter as a function of the approximate infrared fixed point, α_*, or equivalently, the number of massless fermions, N_f, in a vectorial, confining SU(N) gauge theory. We have focused on the crossover region between the walking and QCD-like (non-walking) regimes. Our results show that $\hat{S}$ and also S decrease significantly as α_* decreases in this range. This trend agrees with earlier indications of a decrease in S in walking gauge theories. We have discussed the implications for technicolor models.

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VIII. APPENDIX

In this appendix we present some results on vector meson dominance (VMD) and their relevance to the crossover region that we are studying numerically. In QCD itself, VMD has served as a useful approximate model. The matrix elements for the production of a vector or axial-vector meson from the vacuum are

$$\langle \rho^a | V^b_\mu | 0 \rangle = i\delta^{ab} f_\rho \epsilon_\mu \tag{8.1}$$

$$\langle a^a_1 | A^b_\mu | 0 \rangle = i\delta^{ab} f_{a_1} \epsilon_\mu \tag{8.2}$$

where ϵ_μ are the respective polarization four-vectors. In QCD, $f_\rho \simeq 154$ MeV, so that the first Weinberg sum rule yields $f_{a_1} = \sqrt{f_\rho^2 - f_\pi^2} = 123$ MeV. A pole-dominated form for the current-current correlation functions in the (simplistic) narrow-resonance approximation is

$$\Pi_{V-A}(q^2) = \frac{f_V^2 m_V^2}{m_V^2 - q^2 - i\epsilon} - \frac{f_A^2 m_A^2}{m_A^2 - q^2 - i\epsilon} \ , \tag{8.3}$$

where, for QCD, the ground-state vector and axial-vector mesons are the $\rho(776)$ and $a_1(1230)$, i.e., $V = \rho$ and

$A = a_1$. Using the principal value relation

$$\frac{1}{s - s_0 \mp i\epsilon} = P\left(\frac{1}{s - s_0}\right) \pm i\pi\delta(s - s_0) \ , \qquad (8.4)$$

the pole-dominated form (8.3) gives, for the spectral functions,

$$\rho_V(s) = f_V^2 \delta(s - m_V^2) \qquad (8.5)$$

and

$$\rho_A(s) = f_A^2 \delta(s - m_A^2) \qquad (8.6)$$

Although these are useful simplifications, more realistic analyses of the spectral function sum rules take into account the finite widths of the ρ and a_1 resonances, and also nonresonant contributions [52].

Expanding Eq. (8.3) for large Euclidean q^2, one obtains

$$\Pi_{V-A}(q^2) = -\frac{(f_V^2 m_V^2 - f_A^2 m_A^2)}{q^2} + O\left(\frac{1}{q^4}\right) \ . \qquad (8.7)$$

The asymptotic freedom of the SU(N) gauge theory implies that for $|q^2| \gg \Lambda^2$, $\Pi_{V-A}(q^2) \propto 1/q^4$, so this yields the relation

$$f_V^2 m_V^2 - f_A^2 m_A^2 = 0 \ , \qquad (8.8)$$

which is the second Weinberg sum rule, evaluated in the VMD approximation. Substituting the forms for the spectral functions into Eq. (3.10) yields

$$S_{VMD} = -16\pi \bar{L}_{10} = 4\pi\left(\frac{f_V^2}{m_V^2} - \frac{f_A^2}{m_A^2}\right) \qquad (8.9)$$

Similar substitutions into the first and second Weinberg sum rules yield the relations

$$f_V^2 - f_A^2 = f_P^2 \qquad (8.10)$$

and Eq. (8.8). The narrow-width VMD relations are approximate, since they neglect the sizeable widths of the ρ and a_1, but the Weinberg relations follow from the asymptotic freedom of QCD when one sets $m_u = m_d = 0$.

As N_f increases and there is a crossover in the SU(N) theory from QCD-like behavior to walking behavior, there are resultant changes in how these integral relations are satisfied. For both the QCD-like and walking regimes, the underlying asymptotic freedom of the theory implies that the current-current correlation functions $\Pi_{V-A}(q^2)$ have a $1/q^4$ asymptotic falloff (up to logs) for Euclidean $q^2 \gg \Lambda^2$. However, in a walking regime, the scale of chiral symmetry breaking, given by $f_P \sim \Sigma$, is much less than Λ, and in the interval $\Sigma^2 \ll q^2 \ll \Lambda^2$, $\Pi_{V-A}(q^2)$ have a less rapid falloff, $\sim 1/q^{4-2\gamma}$), where γ is the anomalous dimension of the bilinear fermion operator and is $\gamma \simeq 1$ for strong walking, whence a $\Pi_{JJ}(q^2) \sim 1/q^2$ falloff in this interval. Physically, the coupling α is strong, of order O(1) but slowly varying for an extended interval of energies between Σ and Λ. Thus it is plausible that in a walking theory, rather than just the lowest-lying vector and axial-vector resonances making important contributions to the various spectral function integrals, there could be significant contributions from a number of higher-lying states with the same quantum numbers [39]. In this case, still retaining the VDM approximation, one would generalize the expressions above to be sums over these higher-lying states. For example, the evaluation of Eqs. (3.10), (3.16), and (3.17) in the VMD approximation would read

$$\hat{S}_{VMD} = 4\pi \sum_i \left(\frac{f_{V_i}^2}{m_{V_i}^2} - \frac{f_{A_i}^2}{m_{A_i}^2}\right) \qquad (8.11)$$

$$\sum_i (f_{V_i}^2 - f_{A_i}^2) = f_P^2 \qquad (8.12)$$

and

$$\sum_i (f_{V_i}^2 m_{V_i}^2 - f_{A_i}^2 m_{A_i}^2) = 0 \ . \qquad (8.13)$$

More generally, however, it is not clear that the VMD model would apply in this walking regime. One analytic approximation is to use VMD for a partial evaluation representing contributions from $\sqrt{s} \simeq \Sigma$ and supplement this with a term due to a fermion loop [22]. In the region that we have concentrated on in this paper, in which the theory is crossing over between strong walking behavior and QCD-like behavior, one expects that the saturation of the spectral function integrals has a form that is intermediate between walking and QCD-like. In our actual calculation of $\hat{S}$ via Eq. (3.6), we only need to calculate $\Pi'_{V-A}(0)$, which we do by means of numerical solutions of the relevant SD and IBS equations, so we do not have to deal with questions of how the spectral functions behave in a walking gauge theory. In future work it would be worthwhile to use the connection between these two different ways of calculating S to undersand the hadronic spectrum better in such a walking gauge theory.

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